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THE CRITICAL CASE OF FOURTH-ORDER RESONANCE IN A HAMILTONIAN SYSTEM WITH ONE DEGREE OF FREEDOM†

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The motion of a time-periodic Hamiltonian system with one degree of freedom in the neighbourhood of an equilibrium position is studied. It is assumed that the equilibrium is stable in the first approximation and that fourth-order resonance is present. The critical case is considered, when the system parameters are such that, in order to draw rigorous conclusions about the stability of the equilibrium, terms of order higher than four in the series expansion of the Hamiltonian must be taken into account. Sufficient conditions are derived for stability and instability, and the bifurcations of periodic motions are investigated in the neighbourhood of the equilibrium position when the system parameters pass through values corresponding to the critical case. The results are applied in the problem of the motion of a sphere in a uniform gravity field when there are collisions with the surface of an elliptic cylinder with a horizontal generator. © 1997 Elsevier Science Ltd. All rights reserved.

1. STATEMENT OF THE PROBLEM

Consider a system with one degree of freedom whose motion is described by canonical differential equations with a 2π-periodic time-dependent Hamiltonian $H(x, y, t)$. It is assumed that the origin $x = y = 0$ of the phase space is an equilibrium position, and that the function H is analytic in the neighbourhood of the origin or, at least, that the partial derivatives of H up to a sufficiently high order are continuous.

Suppose that the characteristic exponents of the linearized equations are pure imaginary numbers $\pm i\lambda$ ($\lambda > 0$) and that the system has no resonance of order up to and including three (that is, $k\lambda$ is no an integer for any of $k = 1, 2, 3$), but has a resonance of order four: $4\lambda = N$, where N is an integer.

Using canonical transformations, one can choose the variables x and y so that the first few forms in the series expansion of the Hamiltonian are normalized [1, 2]. Suppose this has already been done. Then, as can be shown by suitable reduction, the Hamiltonian can be written as

$$H = \lambda\tau + \sum_{n=2}^{m} [\gamma_n + \alpha_n \sin(4\chi - Nt) + \beta_n \cos(4\chi - Nt)]\tau^n + O(\tau^{m+1}) \tag{1.1}$$

where $x = \sqrt{(2\tau)} \sin\chi$, $y = \sqrt{(2\tau)} \cos\chi$, m is a sufficiently large natural number, α_n, β_n, γ_n are constant coefficients, and the quantity $O(\tau^{m+1})$ is 2π-periodic as a function of t.

Let us assume that γ_2 and $\alpha_2^2 + \beta_2^2$ do not vanish. We perform a canonical change of variables χ, $\tau \to \varphi, r$ by the formulae

$$\begin{gathered}\chi = \tfrac{1}{4} Nt + \chi_* + \sigma[\tfrac{1}{8}(1+\sigma)\pi + \varphi], \quad \tau = \delta r \\ \delta = (\alpha_2^2 + \beta_2^2)^{-1/2}, \quad \sin 4\chi_* = \delta\alpha_2, \quad \cos 4\chi_* = \delta\beta_2, \quad \sigma = \operatorname{sign}\gamma_2\end{gathered} \tag{1.2}$$

This transformation cancels the term linear in r in the new Hamiltonian and eliminates the time in terms up to and including r^m. In the new variables

$$H = (\varkappa - \cos 4\varphi)r^2 + \sum_{n=3}^{m} (c_n + a_n \sin 4\varphi + b_n \cos 4\varphi)r^n + O(r^{m+1}) \tag{1.3}$$

$$\varkappa = \delta|\gamma_2|, \quad c_n = \sigma\delta^{n-1}\gamma_n, \quad a_n = \sigma\delta^n(\alpha_2\beta_n - \beta_2\alpha_n), \quad b_n = -\delta^n(\alpha_2\alpha_n + \beta_2\beta_n) \tag{1.4}$$

The term $O(r^{m+1})$ in (1.3) is 8π-periodic in t.

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Let us consider the approximate system of equations of motion defined by the truncated Hamiltonian $h = (\varkappa - \cos 4\varphi)r^2$, which is obtained from (1.3) by ignoring terms of order greater than two in r. Phase portraits of this system in the y_1, y_2 plane, where $y_1 = \sqrt{(2r)}\cos\varphi$, $y_2 = \sqrt{(2r)}\sin\varphi$, are shown in Fig. 1(a–c) for the cases $0 < \varkappa < 1$, $\varkappa = 1$ and $\varkappa > 1$, respectively.

The asymptotic trajectories in Fig. 1(a) lie on the rays $\varphi = \pm 1/4 \arccos \varkappa + 1/2l\pi$ ($l = 0, 1, 2, 3$). The period of the motion along the closed trajectory in Fig. 1(c) is $2K(k)[(\varkappa + 1)h]^{-1/2}$, where K is the complete elliptic integral of the first kind, $k^2 = 2(\varkappa + 1)^{-1}$, h is the energy integral constant and the maximum and minimum values of r on such a trajectory are $(\varkappa - 1)^{-1/2}h^{1/2}$ and $(\varkappa + 1)^{-1/2}h^{1/2}$, respectively.

It has been shown [2] that when $0 < \varkappa < 1$ the equilibrium position at the origin is unstable, but if $\varkappa > 1$, it is stable, not only in the approximate system with truncated Hamiltonian but also in the full system defined by the Hamiltonian (1.3).

The case $\varkappa = 1$ is critical. In the approximate system with $\varkappa = 1$ all points on the coordinate axes are equilibrium positions and they are all unstable (Fig. 1b). Examples show [2] that terms of order higher than two in r in the Hamiltonian (1.3) may lead to either stability or instability of the origin.

In this paper conditions will be obtained for stability and instability in the critical case $\varkappa = 1$. These conditions may be written as inequalities involving the coefficients γ_n, α_n, β_n ($n \geqslant 3$) of the normalized Hamiltonian (1.1). We will also investigate the problem of bifurcations of periodic motions in the neighbourhood of the origin as the quantity κ passes through its critical value $\kappa = 1$.

2. FURTHER TRANSFORMATION OF THE NORMAL FORM OF THE HAMILTONIAN

We apply a univalent canonical transformation $\varphi, r \to \psi, R$ in (1.3), by the formulae

$$\varphi = \psi + k_1 R + k_2 R^2 + \ldots + k_{m-2} R^{m-2}, \quad r = R \tag{2.1}$$

The constant coefficients k_i in (2.1) may be chosen so that the transformed Hamiltonian will not contain $\sin 4\psi$ in terms of order up to and including m with respect to R. For example, the coefficients k_1 and k_2 must be defined by the equalities $k_1 = -1/4a_3$, $k_2 = -1/4(a_4 + a_3 b_3)$.

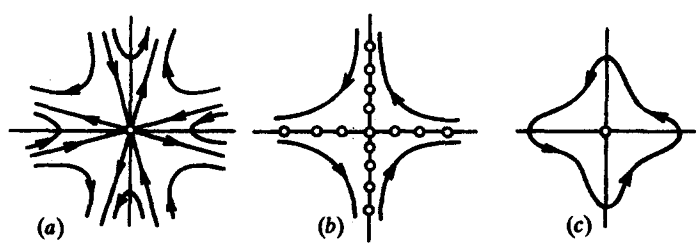
The transformed Hamiltonian (1.3) may be written as

$$H = \sum_{n=2}^{m} \delta_n R^n + (1 - \cos 4\psi) \sum_{n=2}^{m} e_n R^n + O(R^{m+1}) \tag{2.2}$$

The coefficients δ_n and e_n can be expressed in terms of the coefficients of (1.3). In particular, we have $e_2 = 1$ and

$$\delta_2 = \varkappa - 1, \quad \delta_3 = c_3 + b_3, \quad \delta_4 = c_4 + b_4 - \tfrac{1}{2} a_3^2 \tag{2.3}$$

In the critical case the coefficient δ_2 in the first sum of (2.2) vanishes. Suppose that the next coefficient δ_3 is not zero. To investigate the critical and near-critical cases, it will be convenient to perform one further change of variables $\psi, R, t \to \theta, \rho, \eta$, using the formulae



(a) (b) (c)

Fig. 1.

$$\psi = \theta, \quad R = |\delta_3|^{-1}\rho, \quad t = |\delta_3|\eta.$$

In the new variables, the motion is described by canonical equations with the Hamiltonian

$$H = (\varkappa - \cos 4\theta)\rho^2 + [s + k(1 - \cos 4\theta)]\rho^3 + O(\rho^4) \tag{2.4}$$

where $s = \operatorname{sign} \delta_3$, $k = |\delta_3|^{-1} e_3$.

3. STABILITY IN THE CRITICAL CASE

Assuming that $\delta_3 \neq 0$, let us consider the approximate system with Hamiltonian $\Gamma(\theta, \rho)$ obtained from (2.4) by dropping terms of higher than third order of smallness in ρ. The approximate system is integrable and can be fully investigated. For sufficiently small ρ, the phase trajectories of this system in the critical case ($\varkappa = 1$) are shown in Figs 2(b) and 3(b) in the x_1, x_2 plane, where

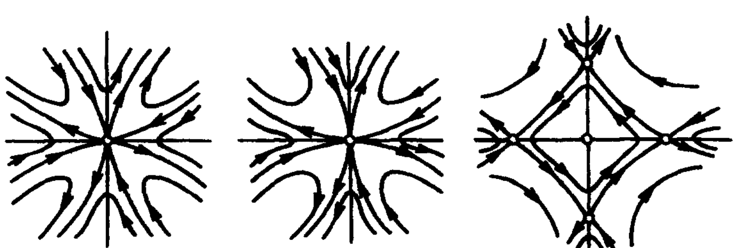
$$x_1 = \sqrt{2\rho}\cos\theta, \quad x_2 = \sqrt{2\rho}\sin\theta \tag{3.1}$$

Figure 2(b) corresponds to the case $s = -1$ ($\delta_3 < 0$) and Fig. 3(b) corresponds to the case $s = 1$ ($\delta_3 > 0$). In the approximate system, the equilibrium position at the origin is unstable when $\delta_3 < 0$ and stable when $\delta_3 > 0$. We will show presently that these conclusions hold not only for the approximate system but also for the system described by the equations with the full Hamiltonian (2.4).

Taking into account that, by (1.4) and (2.3), the expression for δ_3 in terms of the coefficients of Hamiltonian (1.1) is $\delta_3 = \delta^3[\gamma_2\gamma_3 - (\alpha_2\alpha_3 + \beta_2\beta_3)]$, we can formulate the last statement as the following theorem.

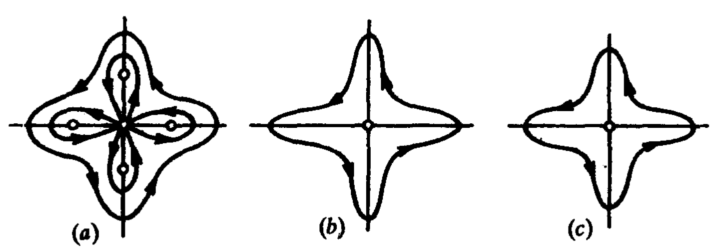
Theorem 1. If the coefficients of Hamiltonian (1.1) are such that $|\gamma_2| = \sqrt{(\alpha_2^2 + \beta_2^2)}$, but at the same time

$$\gamma_2\gamma_3 < \alpha_2\alpha_3 + \beta_2\beta_3 \tag{3.2}$$



(a) (b) (c)

Fig. 2.



(a) (b) (c)

Fig. 3.

then the equilibrium position $x = y = 0$ is unstable; if the inequality obtained from (3.2) by reversing the inequality sign is true, then the equilibrium position is stable.

To prove instability, we define a function V as follows:

$$V = \rho^2 \sin 4\theta \tag{3.3}$$

Taking into account that $\kappa = 1$, while by condition (3.2) we have $s = -1$, we obtain an expression for the derivative of V along trajectories of the equations with Hamiltonian (2.4)

$$dV / d\eta = -4\rho^3[(1 - \cos 4\theta)(2 + O(\rho)) + \rho(3 + O(\rho))] \tag{3.4}$$

For sufficiently small ρ, this function is negative-definite. But since the function V is of fixed sign, it follows by Lyapunov's first instability theorem [3] that the equilibrium is unstable when inequality (3.2) holds.

Now suppose that inequality (3.2) holds with the reverse sign. Then $s = 1$ and $\Gamma(\theta, \rho) + \rho^3 + \nu\rho^2(1 + k\rho)$, where $\nu = 1 - \cos 4\theta$. The approximate system is described by the equations

$$d\theta / d\eta = 3\rho^2 + \nu\rho(2 + 3k\rho), \quad d\rho / d\eta = -4\rho^2(1 + k\rho)\sin 4\theta \tag{3.5}$$

and it has an integral $\Gamma(\theta, \rho) = h = \text{const}$.

We take ρ to be so small that the right-hand side of the first equation in (3.5) is positive and the sign of the right-hand side of the second equation the reverse of that of $\sin 4\theta$ for any k. For small ρ, we have $0 < h \ll 1$.

We now put $h = h_0$ and, considering the equation $\Gamma(\theta, \rho) = h_0$, find the root $\rho = \rho_0(\theta, h_0)$, which describes a phase trajectory of system (3.5) encircling the origin which will "collapse" into the origin as $h_0 \to 0$. The phase trajectory is shown in Fig. 3(b). The maximum and minimum ρ values on this trajectory are given by the expressions

$$\rho_{\max} = h_0^{1/3}, \quad \rho_{\min} = \sqrt{2}h_0^{1/2} / 2 - (1 + 2k)h_0 / 8 + O(h_0^{3/2})$$

If $h = h_0 + \mu(|\mu| \ll 1)$, the root $\rho(\theta, h)$ of the equation $\Gamma(\theta, \rho) = h$ can be represented by a convergent series in powers of μ: $\rho = \rho_0 + \mu\rho_1 + \dots$, where $\rho_1^{-1} = 3\rho_0^2 + \nu\rho_0(2 + 3k\rho_0) > 0$.

Let $I(h)$ be the action variable in the approximate system (3.5) ($I(0) = 0$). Near the phase curve $\rho = \rho(\theta, h)$, $I(h)$ may be expanded in series

$$I = \frac{1}{2\pi}\int_0^{2\pi} \rho(\theta, h)d\theta = I_0 + \mu I_1 + \dots, \quad I_1 = \frac{1}{2\pi}\int_0^{2\pi} \rho_1 d\theta \neq 0$$

where I_0 is $(2\pi)^{-1}$ times the area inside the phase curve $\rho = \rho(\theta, h_0)$. It follows from the inequality $I_1 \neq 0$ that the function inverse to $I(h)$, $h = h(I)$, is analytic in $0 < I \ll 1$.

Considering the full system with Hamiltonian (2.4) (with $\varkappa = 1$ and $s = 1$), if we make the change of variables $\theta, \rho \to w, I$, which transforms the function $\Gamma(\theta, \rho)$ to action-angle variables, we obtain the Hamiltonian of the full system in the form

$$H = h(I) + h_1(w, I, \eta) \tag{3.6}$$

where the function h_1 has period 2π in w and $8\pi\delta_3^{-1}$ in η and is analytic or at least has continuous derivatives to sufficiently high order for $0 < I \ll 1$; moreover $h_1 = o(h(I))$ as $I \to 0$.

Consider the area-preserving mapping of the neighbourhood of the origin defined by the motions of the system with Hamiltonian (3.6) over the period $8\pi\delta_3^{-1}$ of the variable η. A necessary and sufficient condition for the origin to be stable is that, in any arbitrarily small neighbourhood of the origin, a curve encircling the point $I = 0$ exists which is invariant under this mapping [4]. The existence of such curves follows from a theorem of Moser [5] provided the non-degeneracy condition $d^2h/dI^2 \neq 0$ is satisfied.

It can be shown that

$$\frac{d^2h}{dI^2} = \frac{\omega^3}{2\pi}\int_0^{2\pi} \frac{\partial^2\Gamma}{\partial\rho^2}\left(\frac{\partial\Gamma}{\partial\rho}\right)^{-3} d\theta, \quad \frac{1}{\omega} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial\Gamma}{\partial\rho}\right)^{-1} d\theta \tag{3.7}$$

where the integrals are evaluated along a closed trajectory $\rho = \rho(\theta, h)$ of the approximate system (3.5) (Fig. 3b) and ω is the frequency of motion for that trajectory.

Since the quantities $\partial\Gamma/\partial\rho = 3\rho^2 + \nu\rho(2 + 3k\rho)$ and $\partial^2\Gamma/\partial\rho^2 = 6\rho + 2\nu(1 + 3k\rho)$ are positive for sufficiently small ρ, it follows from (3.7) that $d^2h/dI^2 > 0$, and so the non-degeneracy condition is satisfied. This proves the stability statement of Theorem 1.

Remark. Suppose that in the critical case for Hamiltonian (2.2) not only δ_2 but also δ_3 vanish, but $\delta_4 \neq 0$. Then, as in the case $\delta_3 \neq 0$, one can prove that when $\delta_4 < 0$ the origin is unstable and when $\delta_4 > 0$ it is stable. An expression for δ_4 it is stable. An expression for δ_4 in terms of the coefficients of Hamiltonian (1.1) is obtained from formulae (1.4) and (2.3).

Corollary. The results of [2] and of this section imply the following statement about the stability of equilibrium in the presence of fourth-order resonance $4\lambda = N$, which is valid in both critical and non-critical cases.

Theorem 2. If δ_m is the first non-vanishing coefficient in expansion (2.2), the problem of the stability of the equilibrium at $x = y = 0$ is solved by examining the terms of degree up to and including $2m$ in the series expansion of $H(x, y, t)$: if $\delta_m < 0$, the equilibrium is unstable, and if $\delta_m < 0$, it is stable.

4. MOTIONS IN NEAR-CRITICAL CASES

Let us assume that $\delta_3 \neq 0$ and consider the behaviour of the system as the parameter κ passes through its critical value 1. Setting $\varkappa = 1 + \varepsilon(|\varepsilon| \ll 1)$ in (2.4), we obtain the equations of motion

$$\begin{aligned} d\theta / d\eta &= 2(1+\varepsilon-\cos 4\theta)\rho + 3[s + k(1-\cos 4\theta)]\rho^2 + O(\rho^3) \\ d\rho / d\eta &= -4\rho^2(1+k\rho)\sin 4\theta + O(\rho^4) \end{aligned} \tag{4.1}$$

4.1. First let $\delta_3 < 0$ ($s = -1$). As shown in Section 3, in that case when $\varepsilon = 0$ the equilibrium position at the origin is unstable.

Let us consider the approximate system of equations obtained from Eqs (4.1) by omitting quantities that are $O(\rho^3)$ and $O(\rho^4)$. Phase portraits for small ρ are shown in Fig. 2 in the x_1, x_2 plane, where x_1 and x_2 are defined by (3.1).

In Fig. 2(a) ε is negative. The phase trajectories are obtained by a small deformation of the corresponding trajectories in Fig. 1(a). For example, the asymptotic trajectories in Fig. 2(a) no longer lie on straight lines passing through the origin; they are defined by the equations

$$\theta = \pm[\tfrac{1}{4}\arccos(1+\varepsilon) + (1+k\varepsilon)|\varepsilon|^{-\frac{1}{2}}(2+\varepsilon)^{-\frac{1}{2}}\rho + O(\rho^2)] + l\pi/2 \quad (l = 0,\ 1,\ 2,\ 3)$$

If we let $\varepsilon \to 0$, Fig. 2(a) is transformed into Fig. 2(b), where the asymptotic trajectories are tangent to the coordinate axes.

Figure 2(c) corresponds to the case $\varepsilon > 0$. In the approximate system, besides the origin, there are nearby equilibrium positions ρ_*, θ_*, where

$$\rho_* = \tfrac{2}{3}\varepsilon, \quad \theta_* = \tfrac{1}{2}n\pi \quad (n = 0,\ 1,\ 2,\ 3) \tag{4.2}$$

These points are represented in Fig. 2(c) by singular points of the saddle type. The saddle points are linked by heteroclinic doubly-asymptotic trajectories—separatrices. The direction at which they enter or leave a saddle point makes an angle acrtg $\sqrt{(3\varepsilon/[2(3 + 2k\varepsilon)])}$ with the coordinate axis on which the point lies. The separatrices define a part of the phase space filled out by closed trajectories that encircle the stable equilibrium position $x_1 = x_2 = 0$. These trajectories are obtained by a small deformation from the corresponding phase trajectories of Fig. 1(b). As $\varepsilon \to 0$ the closed trajectories, saddle points and separatrices all "collapse" into the origin, and one gets Fig. 2(b).

It can be shown, using Poincaré's small parameter method, that for sufficiently small ε the saddle points of the approximate system give rise to a periodic solution of the complete system (4.1) that is an analytic function of ε. To this solution there corresponds a motion of the system with the original Hamiltonian (1.1) which is 8π-periodic in t and, as $\varepsilon \to 0$, tends to the equilibrium position $x = y = 0$. Calculations show that the characteristic exponents of this periodic motion for small ε are

$\pm 8\sqrt{2}(3|\delta_3|)^{-1}\varepsilon^{3/2} + O(\varepsilon^{5/2})$. Since one of them is positive, it follows from Lyapunov's theorem of stability in the first approximation [3] that the periodic motion is unstable.

Thus, if $\delta_3 < 0$, then, as the parameter $\varkappa$ passes through its critical value $\varkappa = 1$ from the domain $\varkappa < 1$, the unstable equilibrium position $x = y = 0$ becomes stable and an unstable 8π-periodic motion splits off.

4.2. Now let $\delta_3 > 0$ ($s = 1$). In that case, when $\varepsilon = 0$ the origin is stable.

As in Section 4.1, we first consider an approximate system. Its phase trajectories are defined by the equation $\rho^3 + \nu\rho^2(1 + k\rho) + \varepsilon\rho^2 = h = \text{const}$; they are shown for small ρ in Fig. 3.

In Fig. 3(a) the number ε is negative. The origin is unstable. The approximate system also has equilibrium positions ρ_*, θ_* distinct from the origin. These equilibria are defined by equalities (4.2) with ε replaced by $|\varepsilon|$. The equilibria ρ_*, θ_* are represented in Fig. 3(a) by singular points of the centre type, at which $h = (4/27)\varepsilon^3$. If $(4/27)\varepsilon^3 < h < 0$, the phase trajectories are closed and encircle the centres. At $h = 0$ one has an unstable equilibrium—the origin, and homoclinic trajectories doubly-asymptotic to this equilibrium—separatrices. On them $\rho = -(\varepsilon + \nu)(1 + k\nu)^{-1}$, and the angle θ varies in narrow sectors $\theta - {}^1/_2 n\pi| < \Delta$ ($n = 0, 1, 2, 3$), where $\Delta = 1/4\sqrt{2|\varepsilon|}(1 + O(\varepsilon))$, and the maximum value of ρ is $|\varepsilon|$. At $h > 0$ the phase trajectories are closed curves encircling all the singular points and separatrices. On these trajectories, for small $|\varepsilon|$, the quantity $\rho = \rho(\theta, h)$ differs only slightly from the analogous quantity for the closed curve in Fig. 3(b) with the same h value. If $h > 0$ and the Hamiltonian of the approximate system is transformed to action-angle variables, then for small $|\varepsilon|$ the quantity d^2h/dI^2 will differ only slightly from the analogous quantity in (3.7) and, in accordance with Section 3, will not vanish, that is, for $h > 0$ and small $|\varepsilon|$ the Hamiltonian of the approximate system is non-degenerate.

As $\varepsilon \to 0$, the doubly-asymptotic trajectories and the domains surrounding them in Fig. 3(a) "collapse" into the origin, and at $\varepsilon = 0$ one obtains the phase portrait shown in Fig. 3(b).

The phase portrait of the approximate system for $\varepsilon > 0$ is shown in Fig. 3(c). It is obtained from that of Fig. 3(b) by a small deformation.

It can be shown by Poincaré's method that the centre-type singular points existing in the approximate system when $\varepsilon < 0$ (Fig. 3a) give rise, for sufficiently small $|\varepsilon|$, to a periodic solution of the full system (4.1) which is analytic in ε and, as $\varepsilon \to 0$, tends to an equilibrium position at the origin. Corresponding to this solution is a motion of the initial system with Hamiltonian (1.1) which is 8π-periodic as a function of t. Calculations show that, for suitably chosen perturbations q and p, the Hamiltonian of the perturbed motion for this periodic motion admits of a normal form

$$H = \tfrac{1}{2}\Omega(q^2 + p^2) + \tfrac{1}{4}c(q^2 + p^2)^2 + O((q^2 + p^2)^{5/2})$$
$$\Omega = (8/3)\sqrt{2}|\varepsilon|^{3/2}(1 + O(\varepsilon)), \quad c = -(68/3)(1 + O(\varepsilon))$$

If $|\varepsilon|$ is sufficiently small, c does not vanish. Hence, it follows from Moser's invariant curve theorem [5] that the periodic motion is stable.

Another corollary of Moser's theorem and the non-degeneracy of the Hamiltonian of the approximate system when $h > 0$ (as noted above) is that, irrespective of the fact that when $\varepsilon < 0$ the equilibrium position $x = y = 0$ is unstable, the trajectories of the motion in its neighbourhood are bounded: if a trajectory begins close enough to the origin, as the motion continues the function $\rho(t)$ will not exceed a quantity of the same order as $|\varepsilon|$ (recall that $|\varepsilon|$ is the maximum value of ρ on the doubly-asymptotic trajectories of Fig. 3a).

As the parameter $\varkappa$ passes through the critical value $\varkappa = 1$ from the domain $\varkappa < 1$ to the domain $\varkappa > 1$, the 8π-periodic motion (with respect to t) shrinks to the origin at $\varkappa = 1$ and disappears when $\varkappa > 1$, while the origin itself becomes stable.

5. EXAMPLE. THE MOTION OF A SPHERE ON THE SURFACE OF AN ELLIPTICAL CYLINDER

Let us consider the motion of a sphere in a gravity field over a motionless, absolutely smooth open channel shaped like an elliptical cylinder with horizontal generator. As the sphere moves, it collides from time to time with the surface of the channel. We will assume that the collisions are absolutely elastic. The motion may be considered relative to a fixed system of coordinates, $\xi\eta\zeta$ whose η axis is vertical and whose ζ axis points along the generator. We will assume, without loss of generality, that the projection of the velocity of the centre of the sphere onto the ζ axis, which is constant throughout the motion, is equal to zero. Ignoring the dimensions of the sphere, we arrive at the problem of the plane motion of a point mass over an arc of an ellipse $\xi^2a^{-2} + (\eta - b)^2b^{-2} = 1$ [7].

A periodic motion of the point exists in which its trajectory lies on the vertical η, and the point itself, owing to collisions with an arc of the ellipse at the origin $\xi = \eta = 0$, periodically jumps to a certain height l. The period of this motion equals the time $2\sqrt{(2l/g)}$ between two consecutive collisions.

Sufficient conditions have been obtained for this periodic motion to be orbitally unstable and iso-energetically orbitally stable. In particular, it has been shown that there is a fourth-order resonance in the α, β plane, where $\alpha = a^2b^{-2}$, $\beta = lb^{-1}$, on the ray $\beta = 1/4\alpha$. The points c (5, 5/4) and d (10, 5/2) divide this ray into stable and unstable parts (in the notation of this paper, one has $\varkappa = 1$ at these points). When $0 < \alpha < 5$ or $\alpha > 10$, one has orbital stability (here $\varkappa > 1$), while if $5 < \alpha < 10$, the periodic motion is orbitally unstable (here $\varkappa < 1$).

Let us consider the stability of the periodic motion at the critical points c and d. To do this, we fix an energy level equal to mgl and, following [7], use Poincaré's section surface method to construct an area-preserving mapping of the ξ, p_ξ plane into itself (where p_ξ is the momentum corresponding to the ξ coordinate). The periodic motion of the sphere is represented by a fixed point $\xi = 0, p_\xi = 0$ of the mapping. In the neighbourhood of that point the mapping is analytic. By using a real analytic canonical transformation $\xi, p_\xi \to u, v$ it has been reduced to normal form up to and including terms of order six. In complex conjugate variables $\zeta = u - iv$, $\bar\zeta = u + iv$, the normalized mapping may be written as

$$\zeta_1 = i\zeta + f_{21}^*\zeta^2\bar\zeta + f_{03}^*\bar\zeta^3 + f_{32}^*\zeta^3\bar\zeta^2 + f_{50}^*\zeta^5 + f_{14}^*\zeta\bar\zeta^4 + O_7 \tag{5.1}$$

We will not dwell here on the rather cumbersome procedure for calculating the coefficients f_{kl}^*.

Corresponding to the mapping (5.1) (as for any analytic area-preserving mapping) we have [1] a Hamiltonian $H(\zeta, \bar\zeta, t)$, 2π-periodic in t, of class c_∞ which, if not analytic, is such that over the time 2π the transformation realized by the motions of the system with that Hamiltonian is identical with (5.1). It can be shown that the Hamiltonian H can be chosen as follows ($4\lambda = 1$):

$$H = i\lambda\zeta\bar\zeta + \gamma_{22}(\zeta\bar\zeta)^2 + \gamma_{40}e^{-it}\zeta^4 + \gamma_{04}e^{it}\bar\zeta^4 + \gamma_{33}(\zeta\bar\zeta)^3 + \gamma_{51}e^{-it}\zeta^5\bar\zeta + \gamma_{15}e^{it}\zeta\bar\zeta^5 + O_8 \tag{5.2}$$

where the coefficients γ_{kl} do not depend on t. For brevity, we omit their expressions in terms of the coefficients of the mapping (5.1).

After the canonical change of variables $\zeta = -i\sqrt{(2\tau)}e^{i\chi}$, $\bar\zeta = i\sqrt{(2\tau)}e^{-i\chi}$, Hamiltonian (5.2) takes the form of (1.1). Calculations have shown that the coefficients $\alpha_n, \beta_n, \gamma_n$ ($n = 2, 3$) at the point c are as follows: $\alpha_2 = 0$, $\beta_2 = -\gamma_2 = 5/(64\pi)$, $\alpha_3 = 0$, $\beta_3 = -5/(512\pi)$, $\gamma_3 = -49/(1536\pi)$, and those at the point d are: $\alpha_2 = 0$, $\beta_2 = \gamma_2 = 5/(32\pi)$, $\alpha_3 = 0$, $\beta_3 = 45/(128\pi)$, $\gamma_3 = 89/(384\pi)$.

It follows from condition (3.2) that at the critical point c the periodic motion of the sphere along the vertical is iso-energetically orbitally stable, but at d it is unstable.

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